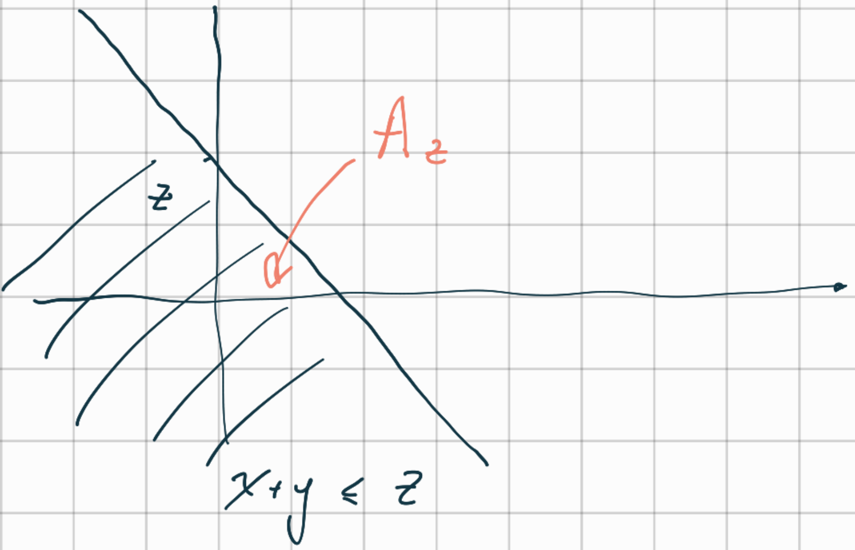
Math 3235 Probability Theory

3/14/23

X_1 and X_2 are 2 uniform $[0,1]$ r.v., independent.

Compute the p.d.f. of $X+Y=Z$

$$\mathbb{P}(X+Y \le z) = F_Z(z)$$



$$\mathbb{P}(X+Y \le z) = \mathbb{P}((X,Y) \in A_z) =$$

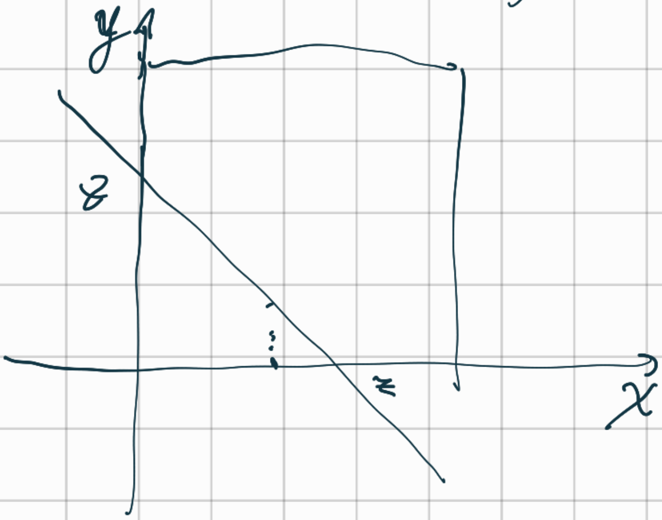
$$\int_{A_z} f(x,y)\,dx\,dy =$$

$$\int_{x+y<z} f(x,y)\,dx\,dy$$

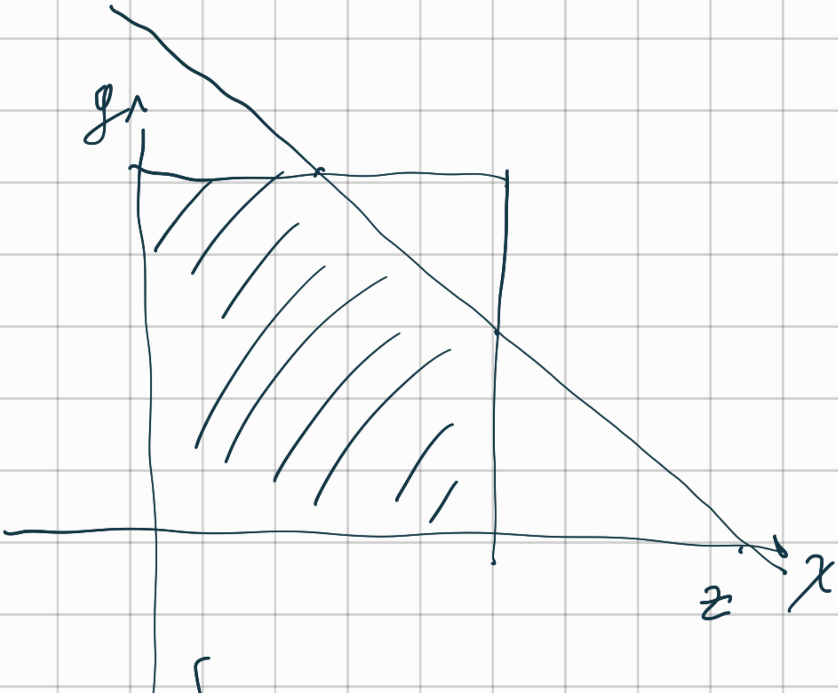
X, Y uniform and indep.

$f(x,y) = 1 \quad 0 \leq x, y \leq 1$

$$\mathbb{P}(X+Y \leq z) = \int_0^z \int_0^{z-x} 1 \, dy \, dx$$



If $z \leq 1$ $\quad \mathbb{P}(X+Y \leq z) = \frac{z^2}{2}$

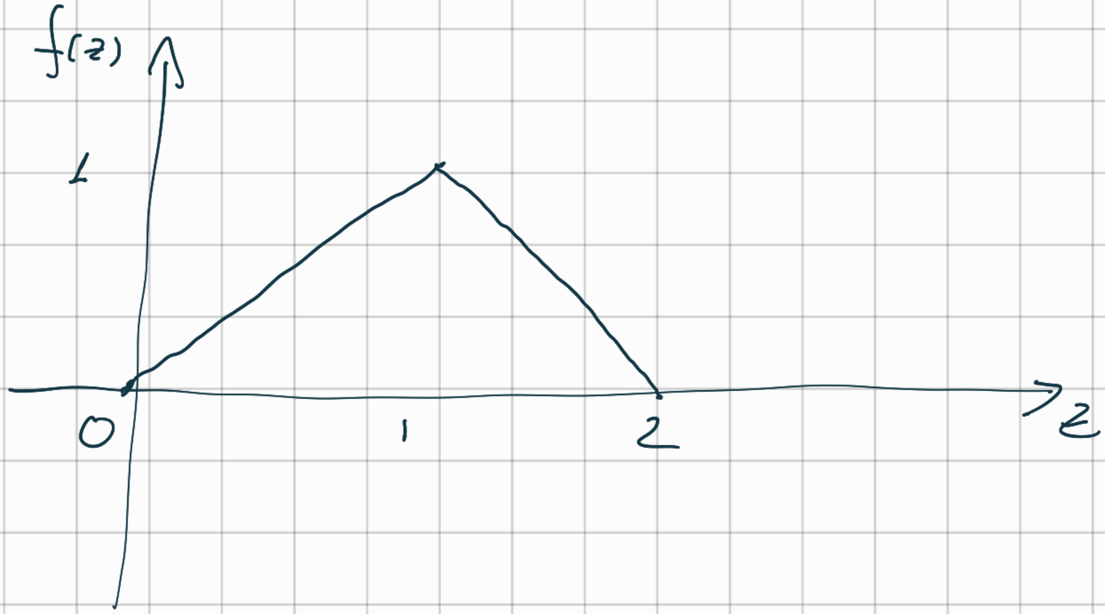


If $z > 1$ $\quad 1 - \frac{(2-z)^2}{2}$

$$z = 1 \qquad \frac{z^2}{2} = \frac{1}{2}$$

$$1 - \frac{(2-z)^2}{2} = \frac{1}{2}$$

$$f_Z(z) = \begin{cases} z & z \leq 1 \\ 2-z & z \geq 1 \end{cases}$$



X Y are exponential 1

and independent

$$f_X(x) = e^{-x} \qquad f_Y(y) = e^{-y}$$

$$f_{X,Y}(x,y) = e^{-(x+y)}$$

$$\mathbb{P}(X+Y \leq z) =$$

$$\iint\limits_{\substack{x+y<z \\ x,y>0}} e^{-(x+y)}\,dx\,dy =$$

$$= \int_0^z dx \int_0^{z-x} dy\, e^{-(x+y)} =$$

$$= \int_0^z dx\, e^{-x} \int_0^{z-x} e^{-y}\,dy =$$

$$= \int_0^z dx\, e^{-x}\left(1 - e^{-(z-x)}\right) =$$

$$= \int_0^z dx\, e^{-x} - \int_0^z e^{-z}\,dx =$$

$$= 1 - e^{-z} - e^{-z}\int_0^z dx =$$

$$= 1 - e^{-z} - ze^{-z}$$

$$F_Z(z) = 1 - e^{-z} - ze^{-z}$$

$$f_Z(z) = e^{-z} - e^{-z} + ze^{-z} = ze^{-z}$$

$X \quad Y$ are exp. λ

indep $f_X(x) = \lambda e^{-\lambda x}$

$$f_{X+Y}(z) = \lambda^2 z e^{-\lambda z}$$

$X_1 \quad X_2 \quad X_3 \quad \ldots \quad X_N$ exp λ ind.

$$Z = \sum_{i=1}^{N} X_i$$

$$f_Z(z) = \frac{\lambda^N}{(N-1)!} z^{N-1} e^{-\lambda z} \qquad \text{Gamma}(N, \lambda)$$

$$= \frac{\lambda^N}{\Gamma(N)} z^{N-1} e^{-\lambda z}$$

$$\Gamma(N) = \int_0^\infty z^{N-1} e^{-z} \, dz$$

We say that X is a Gamma(α, β) r.v. if the p.d.f. of X

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\alpha x}$$

$$\mathbb{P}(X > Y) =$$

$$\iint_{x>y} f(x,y)\,dx\,dy$$

We say that X Y are indep. if

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

$$F_{X,Y}(x,y) = F_X(x)\, F_Y(y)$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}\left[F_X(x)\, F_Y(x)\right] = \frac{\partial}{\partial x} F_X(x) \cdot \frac{\partial}{\partial y} F_Y(y)$$

$$= f_X(x)\, f_Y(y)$$

$$F_{XY}(x,y) = \int_{-\infty}^{x} dz \int_{-\infty}^{y} dw \, f(z,w) =$$

$$= \int_{-\infty}^{x} f_X(z)\, dz \int_{-\infty}^{y} f_Y(w)\, dw =$$

$$= F_X(x)\, F_Y(y)$$

$f_{X,Y}(x,y)$ is the joint p.d.f.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$ marginal

Conditional p.d.f.

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

It follows

a) $f_{X,Y}(x,y) = f_{X|Y}(x|y)\, f_Y(y)$

b) $X \perp\!\!\!\perp Y \implies f_{X|Y}(x|y) = f_X(x)$

Note: $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$f_{X,Y}(x,y) \geq 0$

it follows that if

$$f_X(x) = 0 \implies f_{X,Y}(x,y) = 0 \quad \forall y$$

Thus $f_{X|Y}(x|y)$ is well defined

$$f_{X+Y}(z) = \int_{x+y=z} f_{X|Y}(x|y)\, f_Y(y)\,dy$$

$$= \int_{-\infty}^{\infty} f_{X|Y}(z-y|y)\, f_Y(y)\,dy$$

$$= \int_{-\infty}^{\infty} f_{X,Y}(z-y,y)\,dy$$

if X and Y are indep. and

$Z = X + Y$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)\, f_Y(y)\, dy$$

f_Z is the convolution of f_X and f_Y

$$f_Z = f_X * f_Y$$